On the Stability of Falling Films—Periodic, Finite-Amplitude Waves

An analytical, theoretical investigation regarding periodic finite-amplitude (nonlinear) waves in a Newtonian, isothermal liquid film flowing along a vertical wall is presented. The periodic solution found provides full information regarding stability, wavelength, wave velocity, and wave amplitude as expressed exclusively in terms of physical constants and flow rate.

Runo Barrdahl
Department of Chemical Engineering
Lund Inst. Technology
S-22100 Lund, Sweden

SCOPE

Falling film column equipment is widely used in chemical, process, and other industries for mass and heat transfer purposes. Under operating conditions the liquid films on such columns are in general not planar like the laminar falling film, for which the well-known semiparabolic theoretical solution is due to Nusselt. On the contrary, the film surface is rippled and the ripples propagate along the film. Under such conditions the transfer rate in a mass or heat transfer process is increased considerably beyond the rate predicted by a theoretical model based on the laminar falling film solution generally employed in such modeling. Such an underestimation of transfer rates is unsatisfactory.

The motion of the wavy falling film has also been studied extensively for basic research purposes both theoretically and experimentally. At present the theoretical and experimental results are not in agreement. Most theoretical work leaves as an open question whether the waves are periodic or irregular, but some approximate theories present periodic solutions. A few theoretical works only discuss a criterion for the assertion of stable states of motion, but even at this point the criteria used differ from each other and most authors do not discuss stability considerations at all.

Reports on experimental work carried out often refer to the difficulties encountered in producing a qualitatively satisfactory film (completely symmetric column and nozzle surface, a film without side-wall effects or rotational asymmetry in the case of a cylindrical film, a completely vertical film, a film free from exterior disturbances such as vibrations and air movements, etc.). Thus the question arises as to whether completely periodic solutions really exist and whether deviations from periodicity in practice are due to nonideal experimental conditions, or whether the waves are inherently irregular. Further, some experimental work predicts the existence of a critical Reynolds number below which wavy flow would be unstable, in much the same way as the critical Reynolds number used for the onset of turbulence. Finally, several experimental investigators report the existence of a "short," "fast" wave of "small" amplitude near the entrance region and a "long," (approximately 2-4 times longer than the short wave), "slow" wave of extraordinarily "large" amplitude following a transition of the former wave at a distance typically around 10 cm below the entrance region. Such a large-amplitude wave has not yet been substantiated in theory by analytical methods, although it has been substantiated in fact by numerical methods. The problem of the theoretical approach is most probably due to the difficulties encountered in solving the nonlinear differential equations involved.

In view of the above, it was considered that an analytical, theoretical study of the problem of the stability of falling films might throw light on the physically possible motion. Thus the object of the present work is to elucidate the points discussed above and to provide a full theoretical solution to the problem regarding finite-amplitude waves in a Newtonian, isothermal liquid film flowing along a vertical wall.

CONCLUSIONS AND SIGNIFICANCE

The underlying equations are found to allow a solution completely periodic with respect to the time and the flow direction coordinate. Strictly, the laminar solution due to Nusselt is found to be unstable with respect to the one derived here. Thus it can be inferred that the Reynolds number, although entering into the mathematic formalism, may not be considered as a qualitative measure for a transition from a purely laminar to a convective state of motion, but rather as a component in a larger group of dimensionless numbers determining the quantitative extent of the convection.

The fluid motion is determined completely and uniquely. Wave velocity, wavelength, and wave amplitude are expressed exclusively as functions of: g, the gravitational coefficient; ρ, the mass density; γ, the surface tension; and ν, the kinematic viscosity of the liquid together with Q, the flow rate (or equivalently, D_o, the mean film thickness). Although the total wave amplitude obtained is relatively small (at the most approximately 20% of the mean film thickness) the wavy solution obtained has to be considered as being of a nonlinear character, since all amplitude coefficients (in the Fourier components) are interdependent.

The range of validity of the present solution is limited to conditions in which the product of dimensionless groups $R_o^4 (\rho \nu^2 / \gamma \delta)$ is less than approximately 0.12 (or the dimensionless number T, given below is less than approximately 1/3), where R_o constitutes the would be laminar Reynolds number $R_o = U_o \delta / \nu$. As a consequence, the condition results that R_o in general is restricted to very low values (for example R_o less than approximately 5 for pure water at 20°C). The present solution is thus limited to low values of the Reynolds number.

It may easily be verified that this limitation also applies to the theoretical solutions developed on this subject by others. This in turn means that the experimental results available are out of range (the flow rates are too high). Thus, discrepancies between theoretical and experimental results available must be expected.

Previous Work

Present-day knowledge of the wavy or rippled motion in a falling film must still be considered as unsatisfactorily incomplete for the following reasons. The theory is too approximate to provide sufficient accuracy in comparison with experimental results. This is the case with the work of Kapitza (1965) and subsequent work founded on the assumptions made by Kapitza, such as the work of Shkadov (1967), Gollan and Sideman (1969), Filipescu-Teculescu (1974), and Nakoryakov and Alekseyenko (1981).

First, Kapitza assumed velocity profiles of Fourier components in the perturbed part of the solution to be self-similar with the unperturbed part, i.e., the semiparabolic laminar velocity profile due to Nusselt. This assumption was made in order to obtain a manageable differential equation in time and longitudinal flow coordinate on averaging over the transverse flow coordinate, otherwise in the general case such an equation could not be obtained. Second, in deriving the expression for the wavelength, wave velocity, and wave amplitude Kapitza assumed the wave amplitude to be independent of wavelength and wave velocity (in a first approximation). Both these assumptions constitute unjustified restrictions imposed on the equations, whence a very good agreement with experiments cannot be expected. Even though the work of Kapitza has been improved and refined by the above-mentioned and other workers over the years, the inherent deficiencies due to the assumptions made remain in quality. The present work constitutes an improvement over the work of Kapitza and subsequent work based on his assumptions, in that those assumptions have been abandoned in order to provide a more accurate analytical solution.

The overwhelming majority of works do not involve the use of a stability criterion, which as expressed by Berbente and Ruckenstein (1968) and Tsvelodub (1981) in itself would constitute a rather complicated question, whence the solutions in this case lack completeness since they are not uniquely determined. The present work constitutes an improvement over these works in that it provides a stability criterion and thus a more complete solution. Thus, as will appear below, the present solution provides a result in terms of physical constants and the flow rate only.

Recently, numerical investigations have been carried out on the behavior of the falling liquid film, of which a few examples are the works of Tougou (1981), Tsvelodub (1981), Schlang and Sivashinsky (1982), and Pumir et al. (1983). The present work constitutes a qualitatively different approach to the problem of the stability of the falling liquid film in that it is entirely of an analytical character. The present work thus constitutes a qualitative improvement over these numerical works in that it provides explicit functional relationships, even if it encompasses a much smaller region.

For a more complete list of the large and increasing number of publications in this field—it would lead beyond the scope of the present study to provide an appropriate account of all of them—the reader is referred to the recent literature survey by Barrdahl (1984).

Theory

It is assumed that it is sufficient to consider only two space coordinates, namely the longitudinal flow coordinate, and one transverse coordinate, the film coordinate. Thus the transverse coordinate which is orthogonal to the normal of the wall has been disregarded.

The equations of motion, the Navier-Stokes equations, then become:

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial x} + \nu \cdot \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + g \quad \text{(1a)}$$

$$\frac{\partial v}{\partial t} + u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial y} + \nu \cdot \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \quad \text{(1b)}$$

The equation of continuity becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

The following boundary conditions are commonly considered to be pertinent:

The liquid velocity vanishes at the wall:

$$u = 0; \quad y = 0 \tag{3a}$$

$$v = 0; \quad y = 0 \tag{3b}$$

The film surface is free of stresses:

$$\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \cdot \left[1 - \left(\frac{\partial D}{\partial x}\right)^2\right] = 4 \cdot \frac{\partial D}{\partial x} \cdot \frac{\partial u}{\partial x}; \quad y = D \tag{4a}$$

$$P = -\gamma \frac{\partial^2 D}{\partial x^2} \cdot \left[1 + \left(\frac{\partial D}{\partial x}\right)^2\right]^{-3/2} - 2 \cdot \mu \frac{\partial u}{\partial x} \cdot \left[1 + \left(\frac{\partial D}{\partial x}\right)^2\right] \cdot \left[1 - \left(\frac{\partial D}{\partial x}\right)^2\right]^{-1}; \quad y = D \tag{4b}$$

The kinematic surface condition is:

$$v = \frac{\partial D}{\partial t} + u \cdot \frac{\partial D}{\partial x}; \quad y = D \tag{5}$$

The expression for the mass flow rate is:

$$Q = \rho \int_0^D u \, dy \tag{6}$$

Finally, the expression for the dissipation function, Ξ, is:

$$\Xi = \mu \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 4 \cdot \left(\frac{\partial u}{\partial x}\right)^2\right] \tag{7}$$

Next, the system of Eqs. 1a and 1b is cross-differentiated; i.e., Eq. 1a is differentiated with respect to y and Eq. 1b with respect to x, whereupon the two equations are combined by eliminating the pressure terms. Using Eq. 2 the result is then:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + v \cdot \left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right] - u \cdot \left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right] = \nu \left[\frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 v}{\partial y^2 \partial x} - \frac{\partial^3 v}{\partial x^3}\right] \tag{8}$$

Similarly, Eq. 1a is differentiated with respect to x, Eq. 1b with respect to y, and the resulting equations are added. After making use again of Eq. 2 the result is:

$$-\frac{1}{\rho} \nabla^2 P = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right] \tag{9}$$

where ∇^2 signifies the Laplacian operator $[(\partial^2/\partial x^2) + (\partial^2/\partial y^2)]$.

We now seek a solution of the following form:

$$\Psi(x, y, t) = \sum_{\beta=-N}^{\beta=+N} \zeta_\beta(x)\, \xi_\beta(t)\, F_\beta(y) \tag{10}$$

$$D(x, t) = \sum_{\beta=-N}^{\beta=+N} \zeta_\beta(x)\, \xi_\beta(t)\, D_\beta \tag{11}$$

$$P(x, y, t) = \sum_{\beta=-N}^{\beta=+N} \zeta_\beta(x)\, \xi_\beta(t)\, \Pi_\beta(y) \tag{12}$$

where N is an arbitrarily large number,

$$\zeta_\beta(x) = e^{i(\beta\omega)x} \tag{13}$$

$$\xi_\beta(t) = e^{i(\beta\omega k)t} \tag{14}$$

and the stream function $\Psi(x, y, t)$ is defined by Eq. 2 in the usual manner as follows:

$$u = \frac{\partial \Psi}{\partial y} \tag{15}$$

$$v = -\frac{\partial \Psi}{\partial x} \tag{16}$$

Insertion of expressions 10 to 16 into Eq. 8 and suppression of $\zeta_\beta \xi_\beta$ yields the following $2N + 1$ equations:

$$F_\beta^{(4)}(y) - \left\{2(\beta\omega)^2 + \frac{i(\beta\omega)}{\nu}[k + F_o'(y)]\right\} F_\beta''(y) + \left\{(\beta\omega)^4 + \frac{i(\beta\omega)}{\nu} F_o'''(y) + \frac{i(\beta\omega)^3}{\nu}[k + F_o'(y)]\right\} F_\beta(y) = \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} \frac{i(\alpha\omega)}{\nu} \left\{F_{\beta-\alpha}'(y)[F_\alpha''(y) - (\alpha\omega)^2 F_\alpha(y)] - F_\alpha(y)[F_{\beta-\alpha}'''(y) - (\beta-\alpha)^2 \omega^2 F_{\beta-\alpha}'(y)]\right\} \tag{17}$$

Equation 17 is a fourth-order nonlinear differential equation for the stream functions $F_\beta(y)$.

Insertion likewise of expressions 10 to 16 into Eq. 9 and suppression of $\zeta_\beta \xi_\beta$ yields the following $2N + 1$ equations:

$$-\frac{1}{\rho}\left[\Pi_\beta''(y) - (\beta\omega)^2 \Pi_\beta(y)\right] = 2 F_o''(y)(\beta\omega)^2 F_\beta(y) + 2 \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} \left[-(\alpha\omega)(\beta-\alpha)\omega F_\alpha'(y) F_{\beta-\alpha}'(y) + (\alpha\omega)^2 F_{\beta-\alpha}''(y) F_\alpha(y)\right] \tag{18}$$

Equation 18 constitutes a second-order linear differential equation for the pressure functions $\Pi_\beta(y)$.

Next, a solution is sought for the system of Eqs. 17 and 18 on the following assumptions:

a. The liquid film thickness is much smaller than the characteristic length of the wave structure. This may be expressed as $\lambda \gg D_o$ or $(\beta\omega)\, D_o \ll 1$.

b. The characteristic roots of the homogeneous parts of the differential Eq. 17 are not larger than 1, i.e., the liquid velocities are sufficiently small. This may be expressed as $(\beta T) < 1$ where $T = (\omega D_o)\, Re$, Re being the Reynolds number, $Re = u_o D_o / \nu$.

Under these circumstances, a solution is (after making using of Eqs. (3a–b):

$$F_\beta(y) = A_\beta \phi_\beta(y) + B_\beta \psi_\beta(y) + \chi_\beta(y); \quad \beta \neq 0 \tag{19}$$

$$\Pi_\beta(y) = S_\beta + V_\beta\left(\frac{y}{D_o}\right) + 0\,[T^2, (\beta\omega D_o)^2]; \quad \beta \neq 0 \tag{20}$$

$$F_o(y) = \frac{1}{2!}\left(\frac{y}{D_o}\right)^2 A_o + \frac{1}{3!}\left(\frac{y}{D_o}\right)^3 B_o + \chi_o(y) \tag{21}$$

$$\Pi_o(y) = S_o + V_o\left(\frac{y}{D_o}\right) + 0\,[T^2, (\omega D_o)^2] \tag{22}$$

where A_β, B_β, S_β, V_β, A_o, B_o, S_o, and V_o are constants yet to be determined and

$$\phi_\beta(y) = \left\{\frac{1}{2!}\left(\frac{y}{D_o}\right)^2 + \frac{1}{4!}\left(\frac{y}{D_o}\right)^4 \frac{i(\beta\omega k)}{\nu} D_o^2 + \frac{1}{5!}\left(\frac{y}{D_o}\right)^5 \frac{i(\beta\omega)}{\nu} D_o A_o + 0\,[T^2, (\beta\omega D_o)^2]\right\} \tag{23}$$

$$\psi_\beta(y) = \left\{\frac{1}{3!}\left(\frac{y}{D_o}\right)^3 + \frac{1}{5!}\left(\frac{y}{D_o}\right)^5 \frac{i(\beta\omega k)}{\nu} D_o^2 + \frac{2}{6!}\left(\frac{y}{D_o}\right)^6 \frac{i(\beta\omega)}{\nu} D_o A_o + \frac{2}{7!}\left(\frac{y}{D_o}\right)^7 \frac{i(\beta\omega)}{\nu} D_o B_o + 0\,[T^2, (\beta\omega D_o)^2]\right\} \tag{24}$$

$$\chi_\beta(y) = \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} \frac{i(\alpha\omega)}{\nu} D_o \left\{\frac{1}{5!}\left(\frac{y}{D_o}\right)^5 A_\alpha A_{\beta-\alpha} + \frac{2}{6!}\left(\frac{y}{D_o}\right)^6 A_{\beta-\alpha} B_\alpha + \frac{2}{7!}\left(\frac{y}{D_o}\right)^7 B_\alpha B_{\beta-\alpha} + 0\,[T, (\alpha\omega D_o)^2]\right\} \tag{25}$$

Insertion of the argument $y = 0$ into Eqs. 1a–b, making use simultaneously of the boundary condition 3a–b and introducing Eqs. 10 to 16 and 19 to 25 into the equations results in:

$$-\frac{1}{\rho} i(\beta\omega)\, S_\beta + \nu\, B_\beta \frac{1}{D_o^3} = 0; \quad \beta \neq 0 \tag{26}$$

$$-\frac{1}{\rho} V_\beta \frac{1}{D_o} - \nu\, i(\beta\omega)\, A_\beta \frac{1}{D_o^2} = 0; \quad \beta \neq 0 \tag{27}$$

$$\nu\, B_o \frac{1}{D_o^3} + g = 0 \tag{28}$$

$$-\frac{1}{\rho} V_o \frac{1}{D_o} = 0 \tag{29}$$

In order to determine the remaining constants, the boundary conditions, Eqs. 4a–b and 5 are required. For this purpose functional values taken at $y = D$ are expanded around $y = D_o$ in a Fourier series of the form: $Y(D_o + \epsilon) = Y(D_o) + \epsilon\, Y'(D_o) + (\epsilon^2/2)\, Y''\,(D_o) + \ldots$ where ϵ represents terms of the form $\zeta_\beta\, \xi_\beta$ Γ_β, Γ_β being any function. Assume that terms containing third and higher order terms in ϵ may be neglected, i.e., only up to quadratic terms in $\Gamma_\alpha\, \Gamma_{\beta-\alpha}$ are retained. Then the boundary conditions 4a–b and 5 become after insertion of expressions 10 to 16:

$$F_\beta''\,(D_o) + F_o'''\,(D_o)\, D_\beta + \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} D_{\beta-\alpha} F_\alpha'''\,(D_o) + 0\,[(\beta\omega D_o)^2] = 0 \tag{30}$$

$$F_\beta(D_o) + [k + F_o'(D_o)]\, D_\beta + \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} D_{\beta-\alpha}\, F_\alpha'(D_o) = 0 \tag{31}$$

$$\Pi_\beta(D_o) = \gamma\, (\beta\omega)^2 D_\beta - 2\,\mu\, i(\beta\omega)\, F_\beta'(D_o) - \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} \cdot\, [\Pi_\alpha'(D_o)\, D_{\beta-\alpha} + 2\,\mu\, i(\alpha\omega)\, F_\alpha''\,(D_o)\, D_{\beta-\alpha}] + 0\,[(\beta\omega D_o)^2 \tag{32}$$

Insertion of Eqs. 19 to 22 and 26 to 29 into Eqs. 30 to 32, and combining the same to alternatingly eliminate entering constants yields (the argument $y = D_o$ having been suppressed):

$$A_\beta\,[\phi_\beta''\psi_\beta - \phi_\beta\psi_\beta''] = D_\beta\left\{-\frac{B_o}{D_o^3}\psi_\beta + \left[k + \frac{1}{D_o}\left(A_o + \frac{1}{2}B_o\right)\right]\psi_\beta''\right\} + \chi_\beta\psi_\beta'' - \chi_\beta''\,\psi_\beta - \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} D_{\beta-\alpha}\,[A_\alpha(\phi_\alpha'''\,\psi_\beta - \phi_\alpha'\psi_\beta'') + B_\alpha(\psi_\alpha'''\,\psi_\beta - \psi_\alpha'\psi_\beta'')]; \quad \beta \neq 0 \tag{33}$$

$$B_\beta\,[\phi_\beta\psi_\beta'' - \phi_\beta''\,\psi_\beta] = D_\beta\left\{-\frac{B_o}{D_o^3}\phi_\beta + \left[k + \frac{1}{D_o}\left(A_o + \frac{1}{2}B_o\right)\right]\phi_\beta''\right\} + \chi_\beta\phi_\beta'' - \chi_\beta''\,\phi_\beta - \sum_{\substack{\alpha=-N \\ \alpha\neq\beta \\ \alpha\neq 0}}^{\alpha=+N} D_{\beta-\alpha}\,[A_\alpha(\phi_\alpha'''\,\phi_\beta - \phi_\alpha'\phi_\beta'') + B_\alpha(\psi_\alpha'''\,\phi_\beta - \psi_\alpha'\phi_\beta'')]; \quad \beta \neq 0 \tag{34}$$

$$B_\beta = D_\beta\, \gamma\, \frac{i(\beta\omega)^3\, D_o^3}{\nu\,\rho}\{1 + 0\,[(\beta\omega D_o)^2]\}; \quad \beta \neq 0, \tag{35}$$

$$S_\beta = B_\beta \frac{\nu\,\rho}{i(\beta\omega)} \frac{1}{D_o^3}; \quad \beta \neq 0 \tag{36}$$

$$V_\beta = -\,A_\beta i(\beta\omega)\,\rho\,\nu \frac{1}{D_o}; \quad \beta \neq 0 \tag{37}$$

$$B_o = -\,2\,D_o u_o \tag{38}$$

$$A_o = 2\,D_o u_o - D_o^2 \chi_o'' - D_o^2 \sum_{\substack{\beta=-N\\ \beta\neq 0}}^{\beta=+N} D_{-\beta}\,[A_\beta \phi_\beta''' + B_\beta \psi_\beta'''] + 0[(\beta\omega D_o)^2] \quad (39)$$

$$V_o = 0 \quad (40)$$

$$S_o = \sum_{\substack{\beta=-N\\ \beta\neq 0}}^{\beta=+N} D_{-\beta}\left\{-V_\beta \frac{1}{D_o} - 2\,\mu\, i(\beta\omega)\,[A_\beta\,\phi_\beta'' + B_\beta\,\psi_\beta'']\right\} + 0[(\beta\omega D_o)^2] \quad (41)$$

where

$$u_o = \frac{1}{2}\, g/\nu\, D_o^2; \quad (42)$$

Introduce now expressions 23 to 25 and derivatives thereof into Eqs. 33 to 35 and eliminate B_β by combining Eqs. 34 and 35. After some rearrangement and suppression of the indication of neglected terms of $0[(\beta\omega D_o)^2, T^2]$ the result is:

$$\begin{aligned} A_\beta = &- u_o D_\beta\Bigg[(3z+4) + i(\beta T)\left(\frac{24}{5!}z^2 + \frac{1{,}728}{7!}z + \frac{768}{7!}\right)\Bigg] \\ &+ \sum_{\substack{\alpha=-N\\ \alpha\neq\beta\\ \alpha\neq 0}}^{\alpha=+N}\Bigg\{\frac{i(\alpha\omega)}{\nu} D_o\left[\frac{7}{5!}A_\alpha A_{\beta-\alpha} + \frac{4}{5!}A_{\beta-\alpha}B_\alpha + \frac{36}{7!}B_\alpha B_{\beta-\alpha}\right] \\ &- \frac{D_{\beta-\alpha}A_\alpha}{D_o}\left[3 + i(\beta T)\left(\frac{24}{5!}z + \frac{972}{7!}\right) - \frac{6}{4!}i(\alpha T)\right] \\ &- \frac{D_{\beta-\alpha}B_\alpha}{D_o}\Bigg[1 + i(\beta T)\left(\frac{3}{4!}z + \frac{14}{5!}\right) \\ &- i(\alpha T)\left(\frac{3}{4!}z + \frac{1}{3!}\right)\Bigg]\Bigg\}; \quad \beta \neq 0 \end{aligned} \quad (43)$$

$$\begin{aligned} &u_o D_\beta\,\{(z+2) + i(\beta T)[f(z) - \beta^2\Omega]\} \\ &= \sum_{\substack{\alpha=-N\\ \alpha\neq\beta\\ \alpha\neq 0}}^{\alpha=+N}\Bigg\{\frac{i(\alpha\omega)}{\nu}\left[\frac{9}{5!}A_\alpha A_{\beta-\alpha} + \frac{28}{6!}A_{\beta-\alpha}B_\alpha + \frac{40}{7!}B_\alpha B_{\beta-\alpha}\right] \\ &- \frac{D_{\beta-\alpha}A_\alpha}{D_o}\Bigg[1 + i(\beta T)\left(\frac{48}{5!}z + \frac{1{,}332}{7!}\right) \\ &- i(\alpha T)\left(\frac{2}{3!}z + \frac{10}{4!}\right)\Bigg] - \frac{D_{\beta-\alpha}B_\alpha}{D_o}\Bigg[i(\beta T)\left(\frac{5}{4!}z + \frac{18}{5!}\right) \\ &- i(\alpha T)\left(\frac{5}{4!}z + \frac{160}{6!}\right)\Bigg]\Bigg\}; \quad \beta \neq 0 \end{aligned} \quad (44)$$

where

$$z = \frac{k}{u_o} \quad (45)$$

$$f(z) = \left(\frac{48}{5!}z^2 + \frac{3{,}264}{7!}z + \frac{1{,}152}{7!}\right) \quad (46)$$

$$\Omega = \frac{1}{3}\gamma\frac{\omega^2 D_o}{\rho\, u_o^2} \quad (47)$$

Equation 43 is a nonlinear expression for the amplitude factors A_β and Eq. 44 can be regarded as the dispersion relation for the present system. Equations 35, 43, and 44 prescribe the interrelation between $\Omega(\omega)$, $k(z)$, A_β, B_β, D_β and are quite general, though subject to the restrictions imposed above. The linear parts of expressions 43 and 44 are well established in the literature. New is the incorporation of the terms quadratic in amplitude, i.e., terms of the form $C_\alpha C'_{\beta-\alpha}$, C, C' representing A, B, or D.

Consider now the situation, the object of this study, when an isothermal falling liquid film has attained a state of dynamic equilibrium, i.e., when it has flowed for a sufficiently long time and a sufficiently long distance for entrance effects and the like to have disappeared. Under these conditions and on the basis of the assumption that only a finite number of combinations of the entering parameters (wavelength, wave velocity, wave amplitude, and liquid velocity vector for a given flow rate) will provide a maximally stable flow, i.e., satisfy a relevant stability condition, in a lowest order approximation the motion will then be completely periodic. Namely, if that were not the case, a continuous change among the above-mentioned parameters would conserve the stability level constant and all the resulting possible combinations of these parameters would then have to appear with precisely equal probabilities. This is in conflict with experience. Thus ω and k will hereafter be assumed to be real.

In order to limit the mathematical complexity of the problem an additional assumption will also be introduced at this stage, namely that it is sufficient to consider only the first two Fourier components, namely $\beta = \pm 1, \pm 2$, for the calculation of maximally stable states. Since the imaginary part of D_1 (alternatively the real part) may be chosen to be zero, put:

$$D_1 = \mathscr{a}\, D_o \quad (48)$$

Under these circumstances, a solution to equations 35, 43, and 44 is then, for z in the vicinity of -2:

$$\mathscr{a}^2\,[9\,\Omega + h(z)] = \frac{(z+2)}{(3z+4)}[f(z) - 3\Omega]\,\{1 + 0[\mathscr{a}^2, (\omega D_o)^2, (z+2), T^2]\} \quad (49)$$

$$(3z+4)^2\mathscr{a}^2 = T^2\,[\Omega - f(z)] \cdot [f(z) - 4\Omega]\{1 + 0[\mathscr{a}^2, (\omega D_o)^2, (z+2), T^2]\} \quad (50)$$

$$\begin{aligned} A_{\pm\beta} = &- u_o\, D_{\pm\beta}\Bigg((3z+4) \pm i(\beta T) \\ &\cdot \left\{\frac{24}{5!}z^2 + \frac{1{,}728}{7!}z + \frac{768}{7!} + 3[\beta^2\Omega - f(z)]\right\}\Bigg) \\ &\cdot \{1 + 0[\mathscr{a}^2, (\omega D_o)^2, (z+2), T^2]\}; \quad \beta \neq 0 \end{aligned} \quad (51)$$

$$B_{\pm\beta} = u_o\, D_{\pm\beta}\, 3\, i(\beta T)\,\beta^2\,\Omega; \quad \beta \neq 0 \quad (52)$$

$$\begin{aligned} D_{\pm 2} = &\pm\frac{\mathscr{a}^2 D_o}{2iT[f(z) - 4\Omega]}\Bigg\{(3z+4) \pm iT\Bigg[3(\Omega - f_z) \\ &+ \frac{273}{5!}z^2 + \frac{21{,}900}{7!}z + \frac{9{,}072}{7!} + \frac{(z+2)(3z+4)}{2T^2[f(z) - 4\Omega]}\Bigg]\Bigg\} \\ &\cdot \{1 + 0[\mathscr{a}^2, (\omega D_o)^2, (z+2), T^2]\} \end{aligned} \quad (53)$$

where

$$h(z) = \left(\frac{129}{5!}z^2 + \frac{12{,}108}{7!}z + \frac{5{,}616}{7!}\right) \quad (54)$$

From Eqs. 49 and 50 it is apparent that with z in the vicinity of -2, terms containing α^2, $(z + 2)$, and T^2, respectively, are of the same order of magnitude.

In order to determine the solution uniquely, i.e., a stable or preferred mode, a stability condition is then required. Thus the dissipation function, representing the rate of frictional energy developed, is integrated over the liquid film cross section and averaged over one period with respect to time and longitudinal coordinate. It should be noted that except for a positive multiplicating constant (under isothermal conditions), this represents a measure of the mean rate of entropy production, which should be a minimum (Woods, 1975). In addition, the mass flow rate is set up, and it results in:

$$\Xi_o = \mu \int_0^D \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 4\left(\frac{\partial u}{\partial x}\right)^2\right] dy \quad (55)$$

$$Q_o = \rho \int_0^D u\, dy; \quad (56)$$

where the subscript o indicates the average over x and t.

Insertion of Eqs. 10 to 16, 19, 21, 23 to 25, 38 to 42, and 45 into Eqs. 55 and 56 and integrating gives:

$$\Xi_o = \mu \frac{u_o^2}{D_o}\left(\frac{4}{3} + \sum_{\substack{\beta=-2\\ \beta\neq 0}}^{\beta=+2} A_\beta A_{-\beta}\{1 + 0(\alpha, (\omega D_o)^2, (z+2), T)]\}\right) \quad (57)$$

$$Q_o = \rho\, u_o D_o\left(\frac{2}{3} + \sum_{\substack{\beta=-2\\ \beta\neq 0}}^{\beta=+2} D_{-\beta} A_\beta \{1 + 0[\alpha, (\omega D_o)^2, (z+2), T)]\}\right) \quad (58)$$

Equation 58 enables the mean liquid film thickness D_o to be expressed explicitly. Performing this operation, eliminating D_o in Eq. 57 and, inserting expressions 49 to 54 into the resulting equations yields:

$$\Xi_o = \xi_o\left\{1 + \frac{9}{2}\alpha^2(z+2)(3z+4)\left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right]\right\} \cdot \{1 + 0[\alpha, (\omega D_o)^2, (z+2), T]\} \quad (59)$$

$$D_o = \frac{3}{2}\frac{Q_o}{u_o}\frac{1}{\rho}\left\{1 - 3\alpha^2(3z+4)\left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right]\right\}^{-1/3} \cdot \{1 + 0[\alpha, (\omega D_o)^2, (z+2), T]\} \quad (60)$$

where the argument z of f has been suppressed and where

$$\xi_o = g\, Q_o \quad (61)$$

Differentiation of Eqs. 49, 50, and 59 with respect to α^2 provides the equations necessary for the determination of extreme values of Ξ_o. It results when these operations are carried out to the same degree of approximation, order terms this time having been suppressed:

$$4 = T^2\left[-8\Omega + 5f + \frac{1}{\Omega}(\Omega - f)(f - 4\Omega)\right]\frac{\partial \Omega}{\partial \alpha^2} \quad (62)$$

$$(9\Omega + h) + \left[9\alpha^2 + 3\frac{(z+2)}{(3z+4)}\right]\frac{\partial \Omega}{\partial \alpha^2} = \frac{(f - 3\Omega)}{(3z+4)}\frac{\partial z}{\partial \alpha^2} \quad (63)$$

$$\frac{\partial \Xi_o}{\partial \alpha^2} = \frac{9}{2}\xi_o \alpha^2(z+2)(3z+4)\left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right] \cdot \left\{\frac{1}{\alpha^2} + \frac{1}{(z+2)}\frac{\partial z}{\partial \alpha^2} + \left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right]^{-1} \cdot \frac{1}{(f - 4\Omega)}\left[\frac{1}{4} + \frac{(\Omega - f)}{(f - 4\Omega)}\right]\frac{\partial \Omega}{\partial \alpha^2}\right\} \quad (64)$$

where

$$T^2 = \eta\, \Omega \quad (65)$$

$$\eta = 3\frac{\rho\, u_o^4 D_o^3}{\gamma\, \nu^2} \quad (66)$$

In order to look for extrema now, put

$$\frac{\partial \Xi_o}{\partial \alpha^2} = 0 \quad (67)$$

Elimination of the derivatives in expressions 62 to 64 yields:

$$\frac{9}{2}\xi_o(z+2)(3z+4)\left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right] \cdot \left(1 + \frac{\alpha^2(3z+4)}{(z+2)(f - 3\Omega)}\left\{(9\Omega + h) - \frac{\alpha^2}{2T^2\Gamma} \cdot \left[9 + 3\frac{(z+2)}{(3z+4)}\frac{1}{\alpha^2}\right]\right\} - \frac{1}{2}\frac{\alpha^2}{T^2\Gamma}\left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right]^{-1} \cdot \frac{1}{(f - 4\Omega)}\left[\frac{1}{4} + \frac{(\Omega - f)}{(f - 4\Omega)}\right]\right) = 0 \quad (68)$$

where

$$\Gamma = \left[\Omega - \frac{5}{8}f - \frac{1}{8}\frac{1}{\Omega}(\Omega - f)(f - 4\Omega)\right] \quad (69)$$

Rearrangement of Eq. 68, i.e., multiplication with Γ, and disregarding the trivial solution $(z + 2) = 0$, gives:

$$\Omega = \frac{5}{12}f \pm \frac{1}{12}\left\{13f^2 - 18\Omega\frac{(\Omega - f)(f - 4\Omega)}{(3\Omega - f)} \cdot \left[1 - 3\frac{(3\Omega - f)}{(9\Omega + h)}\right] + \frac{3}{2}\Omega(\Omega - f) \cdot \left[1 + \frac{1}{4}\frac{(\Omega - f)}{(f - 4\Omega)}\right]^{-1}\left[1 + 4\frac{(\Omega - f)}{(f - 4\Omega)}\right]\right\}^{1/2} \quad (70)$$

Equation 70 may be solved by iteration, giving the real roots:

$$\Omega_1 = 0.659 f(z) \quad (71)$$

$$\Omega_2 = 0.128 f(z) \quad (72)$$

The second root Ω_2 has to be discarded since it renders the flow imaginary.

It can be verified that the first root $\Omega_1 = 0.659 f(z)$ makes the second derivative of Ξ_o with respect to α^2 positive, thus providing a minimum in rate of frictional energy developed, or in other words a minimum in the rate of entropy production. This root constitutes a solution valid in the immediate neighborhood of $z = -2$. It might be expected that the region of validity is limited by T, such that $T < \approx 1/3$.

As can be readily seen from the result obtained, the assumptions made relative to magnitude of entering parameters and neglected terms are all well justified. In particular, the magnitudes of the third Fourier components $A_{\pm 3}$ and $D_{\pm 3}$ can be expressed as $-(1/3)[(\Omega - f)/(f - 9\Omega)]$ times $A_{\pm 1}$ and αD_o, respectively, only the lowest order terms being withheld. With the value of $\Omega = 0.659 f$ inserted, it results in the factor 0.023, which is indeed a very small number in comparison with 1.

It should be noted that the linear dispersion relation, assumably valid for infinitesimal wave amplitudes, does not provide a correct result since it would give $\Omega = f(z)$. In addition the amplitudes would remain indeterminate. Thus the incorporation of nonlinear terms is necessary and the problem is to be regarded as intrinsically nonlinear.

Results

The complete solution may be expressed in the form:

$$\omega = u_o \left[3 \frac{\Omega_o \rho f(z)}{\gamma D_o} \right]^{1/2} \quad (73)$$

$$\lambda = 2\pi \frac{1}{u_o} \left[3 \frac{\Omega_o \rho f(z)}{\gamma D_o} \right]^{-1/2} \quad (74)$$

$$\Omega = \Omega_o f(z); \quad \Omega_o = 0.659 \quad (75)$$

$$u_o = \frac{1}{2} g/\nu\, D_o^2 \quad (42)$$

$$k = z\, u_o \quad (45)$$

$$z = -2 + \eta \frac{f^2(z)}{(3z+4)} \Omega_o \cdot \frac{(\Omega_o - 1)(1 - 4\Omega_o)[9\Omega_o f(z) + h(z)]}{(1 - 3\Omega_o)} \quad (76)$$

$$f(z) = \frac{48}{5!} z^2 + \frac{3{,}264}{7!} z + \frac{1{,}152}{7!}; \quad f(-2) = \frac{8}{15} \quad (47)$$

$$h(z) = \frac{129}{5!} z^2 + \frac{12{,}108}{7!} z + \frac{5{,}616}{7!}; \quad h(-2) = \frac{64}{105} \quad (54)$$

$$\alpha^2 = \eta \frac{f(z)^3}{(3z+4)^2} \Omega_o (\Omega_o - 1)(1 - 4\Omega_o) \quad (77)$$

$$\eta = 3 \frac{\rho\, u_o^4 D_o^3}{\gamma\, \nu^2} \quad (66)$$

$$D_o = \delta \left\{ 1 - 3\alpha^2 (3z+4) \left[1 + \frac{1}{4} \frac{(\Omega_o - 1)}{(1 - 4\Omega_o)} \right] \right\}^{-1/3} \quad (78)$$

$$Re = R_o \left\{ 1 - 3\alpha^2 (3z+4) \left[1 + \frac{1}{4} \frac{(\Omega_o - 1)}{(1 - 4\Omega_o)} \right] \right\}^{-1} \quad (79)$$

$$\Xi_o = \xi_o \left\{ 1 + \frac{9}{2} \alpha^2 (z+2)(3z+4) \left[1 + \frac{(\Omega_o - 1)}{(1 - 4\Omega_o)} \right] \right\} \quad (59)$$

where Re is the Reynolds number defined as $Re = (u_o D_o/\nu)$; R_o is the corresponding Reynolds number in the laminar theory, defined as $R_o = (U_o \delta/\nu)$, δ being the laminar liquid film thickness, defined as $\delta = (3 Q_o/g [\nu/\rho])^{1/3}$, and U_o, the laminar liquid surface velocity, given by $U_o = g/\nu\, \delta^2/2$.

$$D(x,t) = D_o \Bigg\{ 1 + 2\alpha \cos[\omega(x+kt)] + \frac{\alpha^2 (3z+4)}{T(f - 4\Omega)} \cdot \sin[2\omega(x+kt)] - \frac{\alpha^2}{(f - 4\Omega)} \left[3(\Omega - f) + \frac{273}{5!} z^2 + \frac{21{,}900}{7!} z + \frac{9{,}072}{7!} + \frac{(z+2)(3z+4)}{2T^2(f-4\Omega)} \right] \cdot \cos[2\omega(x+kt)] + 0(D_{\pm 3}/D_o, T^3, \omega D_o) \Bigg\} \quad (80)$$

$$u(x,y,t) = g/\nu [D_o y - y^2/2] - 2u_o \alpha (3z+4) \cdot \left(\frac{y}{D_o}\right) \cos[\omega(x+kt)] + 2u_o \alpha T \cdot \Bigg\{ \left[\frac{24}{5!} z^2 + \frac{1728}{7!} z + \frac{768}{7!} + 3(\Omega - f) \right] \left(\frac{y}{D_o}\right) - \frac{3}{2} \Omega (3z+4) \left(\frac{y}{D_o}\right)^2 + (3z+4) \cdot \left[\frac{1}{3!} \left(\frac{y}{D_o}\right)^3 z + \frac{2}{4!} \left(\frac{y}{D_o}\right)^4 \right] \Bigg\} \cdot \sin[\omega(x+kt)] - u_o T(\Omega - f) \left(\frac{y}{D_o}\right) \cdot \sin[2\omega(x+kt)] - u_o \frac{T^2(\Omega - f)}{(3z+4)} \cdot \Bigg\{ \left[9(3\Omega - f) + \frac{321}{5!} z^2 + \frac{25{,}356}{7!} z + \frac{10{,}608}{7!} + \frac{(z+2)(3z+4)}{2T^2(f-4\Omega)} \right] \left(\frac{y}{D_o}\right) + (3z+4) \cdot \left[-12\Omega \left(\frac{y}{D_o}\right)^2 + \frac{2}{3!} \left(\frac{y}{D_o}\right)^3 z + \frac{1}{3!} \left(\frac{y}{D_o}\right)^4 \right] \Bigg\} \cos[2\omega(x+kt)] + 0(D_{\pm 3}/D_o, T^3, \omega D_o) \quad (81)$$

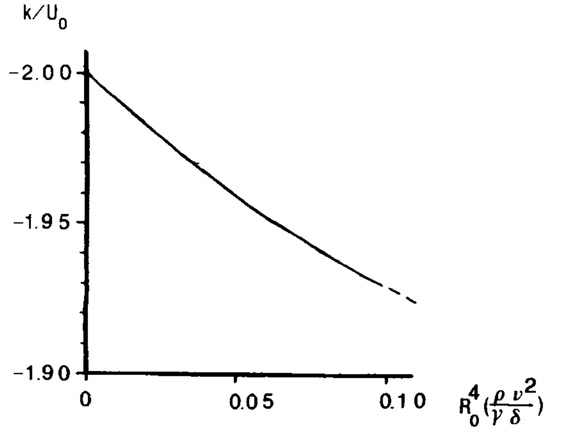

Figure 1. Nondimensional wave velocity k/U_o vs. $R_o^4\,(\rho\,\nu^2/\gamma\,\delta)$.

$$P(x, y, t) = \gamma\,\omega^2 \Big\{ 2\,a \cos\,[\omega(x + kt)] - 4\,\frac{(3z + 4)\,a^2}{T\,(f - 4\Omega)} \sin\,[2\omega(x + kt)] - 4\,\frac{a^2}{(f - 4\Omega)} \Big[3\,(\Omega - f) + \frac{273}{5!}z^2 + \frac{21{,}900}{7!}z + \frac{9{,}072}{7!} + \frac{(z + 2)(3z + 4)}{2\,T^2\,(f - 4\Omega)} \cos\,[2\omega(x + kt)] + 0(D_{\pm 3}/D_o,\, T^3,\, \omega D_o) \Big\} \qquad (82)$$

where only the lowest order terms constituting coefficients for cos and sin have been retained in the expressions.

As a consequence of the fact that all amplitude coefficients are of the same order of magnitude with respect to a^2, $(z + 2)$, and T^2, no completely sinusoidal solution to this problem is permitted, not even in the case of infinitesimal amplitudes (linear theory). Thus the linear approach with a single circular function representing the dependence in the x and t dimensions cannot provide a qualitatively correct result. Although it would be quite possible to obtain, by calculation, additional Fourier terms in expressions 80 to 82 above by means of formulas 35, 43, and 44, these have been omitted since their contribution in the present case is very small, although formally of the same order of magnitude as the first and second Fourier components.

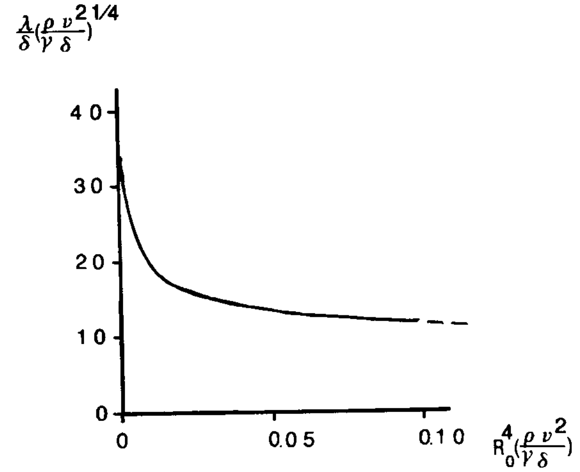

Figure 3. Nondimensional wavelength $(\lambda/\delta)\,(\rho\,\nu^2/\gamma\,\delta)^{1/4}$ vs. $R_o^4\,(\rho\,\nu^2/\gamma\,\delta)$.

A second solution with a longer, slower wave of larger amplitude has not been found during this investigation. As can be seen from Eqs. 49 and 50, valid in the immediate vicinity of $z = -2$ and supposedly being correct in this region, no periodic wave is admitted having an Ω_o less than $1/3$. Thus the search for this supercritical wave will have to continue.

In order to make the functional relationships appear more clearly, the expressions for k, D_o, λ, and a (Eqs. 45, 78, 74, and 77, respectively) may also be put in the form:

$$k = U_o \left\{ -2 + 3\,R_o^4 \left(\frac{\rho\,\nu^2}{\gamma\,\delta}\right) \phi^{11}\,\frac{f^2\,(z)}{(3z + 4)} \cdot \Omega_o\,\frac{(\Omega_o - 1)(1 - 4\Omega_o)}{(1 - 3\Omega_o)}\,[9\Omega_o f(z) + h(z)] \right\} \phi^2 \qquad (83)$$

$$D_o = \delta\,\phi \qquad (84)$$

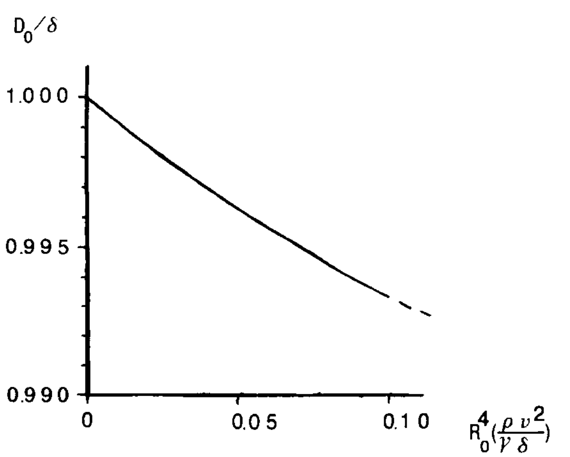

Figure 2. Nondimensional mean liquid film thickness D_o/δ vs. $R_o^4\,(\rho\,\nu^2/\gamma\,\delta)$.

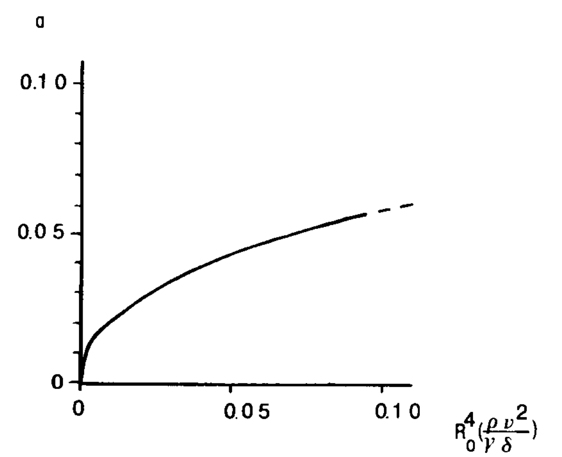

Figure 4. Nondimensional amplitude coefficient a vs. $R_o^4\,(\rho\,\nu^2/\gamma\,\delta)$.

$$\phi = \left\{1 - 3a^2(3z+4)\left[1 + \frac{1}{4}\frac{(\Omega_o - 1)}{(1-4\Omega_o)}\right]\right\}^{-1/3} \quad (85)$$

$$\lambda = \delta\left(\frac{\rho\,\nu^2}{\gamma\,\delta}\right)^{-1/4}\left[R_o^4\left(\frac{\rho\,\nu^2}{\gamma\,\delta}\right)\right]^{-1/4}\frac{2\,\pi}{[3\,\Omega_o f(z)\,\phi^3]^{1/2}} \quad (86)$$

$$a = \left[3\,R_o^4\left(\frac{\rho\nu^2}{\gamma\delta}\right)\phi^{11}\frac{f^3(z)}{(3z+4)^2}\Omega_o(\Omega_o - 1)(1-4\Omega_o)\right]^{1/2} \quad (87)$$

In addition, the condition for convergence that T be $\leqq 1/3$ (or $T^2 \leqq 1/9$) can be expressed as:

$$R_o^4\left(\frac{\rho\,\nu^2}{\gamma\,\delta}\right) \leqq \frac{1}{27}\phi^{-11}\frac{1}{\Omega_o f(z)} \quad (88)$$

Figures 1–4 depict k/U_o, D_o/δ, $\lambda/\delta\,(\rho\,\nu^2/\gamma\,\delta)^{1/4}$, and a, respectively, vs. $R_o^4\,(\rho\,\nu^2/\gamma\,\delta)$, which constitutes the essential parameter in this context. The value of Ω_o has been taken at $z = -2$ for simplicity but this should not introduce any significant errors.

Since the mean liquid film thickness D_o is always smaller than the would-be laminar film thickness δ (ϕ less than unity) for all nonvanishing flow rates, mass is propagated in the direction of motion by the waves under conditions of less frictional energy developed than in the purely laminar flow. The wavy motion is thus more stable than the purely laminar one.

For the sake of comparison, the result of linear theory is: $k = -2\,U_o$, which is equal to the limiting value of the present theory as the flow rate tends to zero. At higher flow rates, the present theory provides a correction resulting in a factor numerically less than -2. The wavelength of linear theory obeys a functional relationship similar to the one in expression 86, however in that expression the last factor in the denominator is replaced by the expression $[3f(-2)]^{1/2}$. With $\Omega_o = 0.659$ the present theory provides a wavelength that is approximately 23% longer than the one of linear theory at low flow rates. At higher flow rates the difference is still larger. a is indeterminate in linear theory, as in all theories not having an appropriate stability criterion.

A comparison with the theory of Kapitza reveals that none of the variables k, D_o, λ, or a obeys the same functional relationships as the present theory. This is also expected because of the approximate character of the Kapitza theory.

Acknowledgment

I wish to express my gratitude to Walter Peier of Zürich, under whose supervision this study began, for all the stimulating and encouraging discussions. I also wish to express my thanks to Åke Jernqvist and Roland Wimmerstedt of Lund for their assistance in making the completion of this work possible.

Notation

A_β = amplitude factor for component β, m^2/s
a = amplitude coefficient
B_β = amplitude factor for component β, m^2/s
D = liquid film thickness, m
D_o = mean liquid film thickness, m
D_β = Fourier component β of liquid film thickness, m
F_β = Fourier component β of stream function, m^2/s
g = gravitational coefficient, m/s^2
k = wave velocity, m/s
P = pressure, $kg/m \cdot s^2$
Q = mass flow rate per unit liquid film width, $kg/m \cdot s$
$Re = u_o D_o/\nu$ = Reynolds number
$R_o = U_o\,\delta/\nu$ = Reynolds number for laminar flow (due to Nusselt)
S_β = amplitude factor for component β, $kg/m \cdot s^2$
$T = \omega\,u_o\,D_o^2/\nu = (\omega D_o)\,Re$
t = time coordinate, s
U = laminar longitudinal velocity (due to Nusselt), m/s
U_o = laminar longitudinal velocity at $y = \delta$, m/s
u = longitudinal velocity, m/s
u_o = mean longitudinal velocity at $y = D_o$, m/s
v = transverse velocity, m/s
v_o = mean transverse velocity at $y = D_o$, m/s
x = longitudinal coordinate, m
y = transverse coordinate, m
$z = k/u_o$ = nondimensional wave velocity

Greek letters

α = integer summation number
β = integer summation number
γ = surface tension coefficient, kg/s^2
δ = laminar liquid film thickness, m
$\eta = 3\,(\rho\,u_o^4\,D_o^3/\gamma\,\nu^2)$
$\lambda = 2\pi/\omega$ = wavelength, m
μ = dynamic viscosity coefficient, $kg/m \cdot s$
Ξ = dissipation function, per unit liquid film width, $kg/m \cdot s^3$
$\xi_o = g\,Q_o$, kg/s^3, Q_o being the mean mass flow rate per unit film width
Π_β = Fourier component of pressure, $kg/m \cdot s^2$
ρ = mass density, kg/m^3
ν = kinematic viscosity coefficient, m^2/s
ϕ_β = first part of solution for stream function
χ_β = inhomogeneous part of solution for stream function
Ψ = stream function, m^2/s
ψ_β = second part of solution for stream function
$\Omega = (1/3)(\omega^2\,\gamma\,D_o/\rho\,u_o^2)$
ω = wave number, 1/m

Literature Cited

Barrdahl, R., "On the Stability of Falling Films—A Literature Survey," LUTKDH/(TKKA-3001)/1-40/(1984), ISSN 0346-8054, Dept of Chemical Engineering I, Lund Inst. Technology, Lund, Sweden (1984).

Berbente, C. P., and E. Ruckenstein, "Hydrodynamics of Wave Flow," *AIChE J.*, **14**(5), 772 (1968).

Filipescu-Teculescu, M., "Miscarea Ondulata a Filmelor Subtiri de Lichid," *Revista de Chimie (Roum)*, **25**(10), 811 (1974).

Gollan, A., and S. Sideman, "On the Wave Characteristics of Falling Films," *AIChE J.*, **15**(2), 301 (1969).

Kapitza, P. L., "Wave Flow of Thin Layers of a Viscous Fluid," *Coll. Papers of P. L. Kapitza*, Pergamon, Elmsford, NY, **2**, 662 (1965).

Nakoryakov, V. Ye., and S. V. Alekseyenko, "Waves on a Film Flowing Down an Incline," *Fluid Mech-Sovjet Res.*, **10**(3), (1981).

Pumir, A., et al., "On Solitary Waves Running Down an Inclined Plane," *J. Fluid Mech.*, **135**, 27 (1983).

Shkadov, V. Ya., "Wave Flow Regimes of a Thin Layer of Viscous Fluid Subject to Gravity," *Izv. AN SSSR Mekh. Zhidk. i Gaza*, **2**(1), 43 (1967).

Schlang, T., and G. I. Sivashinsky, "Irregular Flow of a Liquid Film Down a Vertical Column," *J. Physique*, **43**, 459 (1982).

Tougou, H., "Deformation of Supercritically Stable Waves on a Viscous Liquid Film Down an Inclined Plan Wall with the Decrease of Wave Number," *J. Phys. Soc. Jpn.*, **50**(3), 1,017 (1981).

Tsvelodub, O. Yu., "Steady-State Travelling Waves on a Vertical Liquid Film," *Fluid Mech-Sovjet Res.*, **10**(3), 1 (1981).

Woods, L. C., *The Thermodynamics of Fluid Systems*, Clarendon Press, Oxford, 148 ff, 217 ff (1975).

Manuscript received Jan. 3, 1985, and revision received June 24, 1985.